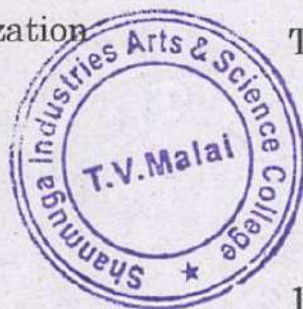


SECTION C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

16. State and prove Hahn-Banach theorem.
17. State and prove Gram-Schmidt orthogonalization process.
18. State and prove Spectral theorem.
19. Prove that $r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}$.
20. If A is a commutative B^* -algebra then prove that the Gelfand mapping $x \rightarrow \hat{x}$ is an isometric $*$ - isomorphism of A onto the commutative B^* -algebra $\mathcal{C}(\mathcal{M})$.



APRIL/MAY 2025

23PMA41 — FUNCTIONAL ANALYSIS

Time : Three hours

Maximum : 75 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL questions.

1. Define normed linear space.
2. State Closed Graph theorem.
3. State Schwarz inequality.
4. Define orthogonal complement.
5. If T is normal then prove that the M_i 's are pairwise orthogonal.
6. Define non-singular.
7. Prove that Z is a subset of S .
8. Prove that $\sigma(x^n) = \sigma(x)^n$.
9. State Gelfand mapping.
10. Define Banach $*$ - algebra.

SECTION B — ($5 \times 5 = 25$ marks)

Answer ALL the questions.

11. (a) Let N and N' be normed linear space and T a linear transformation of N into N' . Then prove that the following conditions on T are all equivalent to one another :

- (i) T is continuous
(ii) T is continuous at the origin in the sense that $x_n \rightarrow 0 \Rightarrow T(x_n) \rightarrow 0$.

Or

- (b) State and prove open mapping theorem.

12. (a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Or

- (b) If $\{e_i\}$ is an orthonormal set in a Hilbert space H , then prove that $\sum |x, e_i|^2 \leq \|x\|^2$ for every vector x in H .

13. (a) Let B be a basis for H , and T an operator whose matrix relative to B is $[\alpha_{ij}]$. Then prove that T is non-singular iff $[\alpha_{ij}]$ is non-singular, and in this case $[\alpha_{ij}]^{-1} = [T^{-1}]$.

Or

- (b) If T is normal then prove that M_i 's span H .
(a) Prove that $\sigma(x)$ is non-empty.

Or

- (b) Prove that the mapping $x \rightarrow x^{-1}$ of G into G is continuous and is therefore a homeomorphism of G onto itself.

15. (a) If f_1 and f_2 are multiplicative functionals on A with the same null space M then prove that $f_1 = f_2$.

Or

- (b) Prove that the following conditions on A are all equivalent to one another :

- (i) $\|x\|^2 = \|x^2\|$ for every x
(ii) $r(x) = \|x\|$ for every x
(iii) $\|\hat{x}\| = \|x\|$ for every x .